One loop calculation of QCD with domain-wall quarks [*]

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We calculate one loop corrections to the domain-wall quark propagator in QCD. We show how the wave function is renormalized in this theory. Especially we are interested in the behavior of the massless fermion mode, which exists near the domain wall at the tree level. We show that this massless mode is stable against the quantum correction.

1. Introduction

The theory of domain wall fermion was originally introduced to treat the chiral gauge theory [1]. Apart from this expectation the domain wall fermion is regarded as a suitable formulation to treat the vector like massless QCD [2]. This is because of the following great advantages comparing with the ordinary Wilson or Kogut-Susskind fermion; (i)The number of flavors is not fixed. (ii)The renormalization of mass is multiplicative. In other words, if a massless mode exists in the tree level it is stable against the quantum correction. The first one is almost trivial but the latter is not. It has been understood from an intuitive discussion and a numerical simulation [2,3]. The aim of this paper is to confirm the latter nature, especially the stability of the massless mode by lattice perturbation theory.

In this paper we set the lattice spacing $a = 1$ and take the $SU(N)$ gauge group with second Casimir C_2. We adopt the domain wall fermion of Shamir type [2] to describe massless quarks. This type of domain wall fermion is a variation of the Wilson fermion with sufficiently many N_S flavors and special form of mass matrix. The only difference from the Wilson fermion action is the fermion bilinear term. In the perturbation theory the gluon propagator and the gauge interaction terms are exactly same as those in the ordinary Wilson fermion perturbation theory [4,5] with N_S flavors. The peculiar thing to the domain wall fermion is the fermion propagator which is given

as an inverse of the fermion bilinear term,

$$\int_{-\pi}^{\pi} \frac{d^4p}{(2\pi)^4} \overline{\psi}(-p) \Big[\sum_{\mu} i\gamma_{\mu} \sin p_{\mu}$$
$$+ W^+(p)P_+ + W^-(p)P_- \Big] \psi(p) \quad (1)$$

where $P_{\pm}$ is a projection operator; $P_{\pm} = (1 \pm \gamma_5)/2$. Our mass matrix $W^{\pm}(p)$ is given as

$$W^+(p) = \begin{pmatrix} -W(p) & 1 & & \\ & & \ddots & \\ & & \ddots & 1 \\ & & & -W(p) \end{pmatrix} \quad (2)$$

$$W^-(p) = (W^+(p))^{\dagger} \quad (3)$$

$$W(p) = 1 - M + \sum_{\mu}(1 - \cos p_{\mu}). \quad (4)$$

In spite of the Dirac mass M in the mass matrix, the above action describes one massless fermion and $N_S - 1$ excited modes at the momentum region $p_{\mu} \sim 0$ by virtue of this matrix form. This means that M is not a physical quark mass but it is rather a cut-off order unphysical mass like Wilson mass. M plays an important role as a parameter of the theory in order to satisfy $|W(p_{\mu} \sim 0)| < 1$, which is needed for the existence of a massless mode.

In the following we will see this massless mode is stable against the quantum correction.

2. Massless Mode

Before calculating the one loop correction we show how the massless mode is guaranteed in the domain wall fermion system.

The easiest way to see the massless mode is to diagonalize the fermion bilinear term. The bilinear term can always be diagonalized by rotating the right and left mode of the Dirac fermion independently by two unitary matrices U and V,

$$\psi_{\mathrm{R}} \to U\psi_{\mathrm{R}}, \qquad \psi_{\mathrm{L}} \to V\psi_{\mathrm{L}}. \qquad (5)$$

These two unitary matrices U and V should be chosen to diagonalize two different hermite mass squared matrices W^+W^- and W^-W^+. For example the mass squared eigenvalues are following at $M = 0.8$ and $N_S = 20$,

$$(1.43516, \quad \cdots \quad 0.645049, \quad 10^{-18}). \qquad (6)$$

The important point is that there is only one massless mode and other modes are excited ones with cut-off order mass. By virtue of this nature the domain wall fermion can be regarded as one massless Dirac fermion system with extra excited modes decoupled.

This property of the mass eigenvalues can be understood from the discussion of the eigenstate equation of the mass squared matrix W^-W^+,

$$(W^-W^+)_{st}\xi_t^i = M_i^2\xi_s^i \qquad (7)$$

where ξ_i and M_i^2 are the i-th eigenstate and eigenvalue. By substituting mass matrix (2) and (3). we have three types of equations depending on s.

$$- W\left(\xi_{s+1}^i + \xi_{s-1}^i\right) + \left(1 + W^2 - M_i^2\right)\xi_s^i = 0 \quad (8)$$
$$-W\xi_2^i + \left(W^2 - M_i^2\right)\xi_1^i = 0 \quad (9)$$
$$-W\xi_{N_S-1}^i + \left(1 + W^2 - M_i^2\right)\xi_{N_S}^i = 0 \quad (10)$$

From the equation (8) which is valid for $2 \le s \le N_S - 1$ we have two kinds of solutions depending on the eigenvalue. When the eigenvalue is small $M_i^2 \le (1 - W)^2$ we have exponential solution,

$$\xi(s) = Ae^{\pm\alpha s}, \quad \cosh\alpha = \frac{1 + W^2 - M_i^2}{2W} \qquad (11)$$

When the eigenvalue is in the region $(1 - W)^2 \le M_i^2 \le (1 - W)^2$ we have an oscillating solution

$$\xi(s) = Ae^{\pm ias}, \quad \cos a = \frac{1 + W^2 - M_i^2}{2W} \qquad (12)$$

And when the eigenvalue is larger; $M_i^2 \ge (1-W)^2$ we again have an exponential solution.

Besides the general equation (8) the solutions have to satisfy the boundary conditions given by the equations (9) and (10),

$$-W\xi_0 + \xi_1 = 0 \qquad (13)$$
$$\xi_{N_S+1} = 0 \qquad (14)$$

Here we assume that N_S is infinitely large. First, a single dumping solution can satisfy the first boundary condition with a dumping factor W. This is nothing but a zero mode solution with $M_i^2 = 0$. Other dumping solutions cannot satisfy this condition. On the other hand oscillating ones can always satisfy (13) with suitable M_i^2. Indeed we can solve the conditions in this case. As a result we have one dumping solution with zero eigenvalue and oscillating solutions with eigenvalues in the region $(1-W)^2 \le M_i^2 \le (1+W)^2$. For example the tree level result at $M = 0.8$ has one small eigenvalue and $N_S - 1$ eigenvalues in the region; $(1 - W)^2 = 0.64 \le M_i^2 \le (1 + W)^2 = 1.44$. This is a good coincidence with the above discussion.

3. One Loop Calculation

Now we calculate the one loop correction to the fermion propagator, which is given by a tadpole diagram with a gluon loop and a half-circle diagram, in which both gluon and fermion runs. We are interested in stability of the massless mode given in the $p_\mu \to 0$ limit. We only require the leading terms in p_μ to see the massless mode.

The contribution from the tadpole diagram is

$$\Sigma^{\mathrm{tadpole}} = g^2 C_2 T \left(\frac{1}{2}i\not{p} + 2\right)\delta_{s,t} \qquad (15)$$

with numerical factor $T = 0.15461$. The first term in (15) is finite, and the second term is cut-off order. We can see that $\Sigma^{\mathrm{tadpole}}$ is diagonal in flavor space.

The correction from the half circle diagram cannot be calculated analytically because of its complicated dependence on the flavor index s, t, which is brought by the fermion propagator. We will evaluate the form of $\Sigma_{s,t}^{\mathrm{half}}$ according to [5] by separating the loop momentum into two region. The logarithmically divergent part can be calculated analytically and other linearly divergent

and finite terms are given by integrating the loop momentum numerically, where we let $N_S = 20$ for the calculation. And finally the correction is written in a simple form,

$$\Sigma^{\text{half}} = -\left[i\not{p} \left(I^{\pm}_{\log} + I^{\pm}_{\text{finite}} \right) P_{\pm} + M^{\pm} P_{\pm} \right] \quad (16)$$

where finite term $I^{\pm}_{\text{finite}}$ and linearly divergent $M^{\pm}$ are complicated functions of s, t and M.

The logarithmic divergence appears only in the wave function and is localized near the boundary

$$I^{+}_{\log} = \frac{1}{16\pi^2} g^2 C_2 M (2 - M)(1 - M)^{s+t-2}$$
$$\times \left(\ln(\pi^2) + \frac{1}{2} - \ln p^2 \right). \quad (17)$$

In the diagonal basis of (5) it is exactly localized in the boundary. This means that the logarithmic divergence can be renormalized into the zero mode wave function. On the other hand, the linear divergence in the mass term should be treated as an additive quantum correction to the mass matrix. This is because the tree level mass matrix is cut-off order and a correction of the same order is always allowed.

In the following we will see whether the massless mode is preserved against the correction to the mass matrix. The contribution from the tadpole diagram is to modify the mass parameter to $M - 2g^2 C_2 T$. Although $M^{\pm}$ has nontrivial flavor dependence, it is much smaller ($\sim 18\%$ at most) than that of the tadpole diagram and it turns out to be a similar form to the tree level mass matrix; minus values along the diagonal line and plus ones along next to the diagonal. Differently from the tree level mass matrix (2), it's off-diagonal elements are non-zero. But they are very small comparing the tadpole contribution ($\sim 7\%$ at most) and dump exponentially along the off-diagonal line with dumping factor 0.12 at $M = 0.8$.

By virtue of the similarity to the tree level mass matrix we may expect that the good nature of the domain wall fermion is preserved at the one loop level. Diagonalizing the effective mass matrix numerically we have following sequence of the mass squared eigenvalues for $M = 0.8$,

$$(2.36322, \quad \cdots \quad 0.306761, \quad 10^{-11}). \quad (18)$$

Here again we have a single zero eigenvalue and $N_S - 1$ cut-off order ones. Corresponding to them we also have one dumping solution and $N_S - 1$ oscillating ones.

From this peculiar distribution of eigenvalues and eigenstates we may expect that the same mechanism works as at the tree level and the above nature is guaranteed. The one loop effective mass matrix is well approximated by the same form of the matrix (2). Actually with the approximated value of $W = 0.5141$ the numerically solved exact results are well reproduced as an eigenvalue problem of this simple matrix. The above W is about 7% smaller than the dumping factor of the exact zero mode solution, which should be coincide in our approximation. This difference may be the effect of the off-diagonal parts which we neglected.

Our conclusion is that the massless mode is preserved at one loop level because the mass matrix has approximately the same form as at the tree level. This fact also guarantees the other modes have cut-off order mass. This approximation is supported by the following reasons; (i) main quantum correction comes from the tadpole diagram, which is diagonal, (ii) the off-diagonal parts are small and dump exponentially off the diagonal line, (iii) numerically solved mass eigenvalues and eigenstates are well reproduced by this approximated form of the mass matrix. For our approximation to work it is essentially important that the off-diagonal parts dump exponentially.

At last we comment that the finite parts in the wave function subtracted with the logarithmic divergence can be diagonalized simultaneously. At $M = 0.8$ the Z-factor of the zero mode is given as $Z = 1 - 0.0216602 g^2 C_2$.

REFERENCES

1. D. B. Kaplan, *Phys. Lett.* **B288** (1992) 342.
2. Y. Shamir, *Nucl. Phys.* **B406** (1993) 90.
3. T. Blum and A. Soni, *Phys. Rev.* **D56** (1997) 174.
4. H. Kawai, R. Nakayama and K. Seo, *Nucl. Phys.* **B189** (1981) 40.
5. L. H. Karsten and J. Smit, *Nucl. Phys.* **B183** (1981) 103.